Math 217 Fall 2025
Quiz 32 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   Let V be a vector space and let $T : V \to V$ be a linear transformation.

   (a) Fix any eigenvalue λ of T. The λ-*eigenspace* of T is ...

   Solution: Let λ be an eigenvalue of T. The λ-eigenspace of T is the subset of V defined by

   $$E_\lambda \ = \ \{\, v \in V : T(v) = \lambda v \,\}.$$

   In other words, it is the set of all eigenvectors of T with eigenvalue λ, together with the zero vector. In fact, it is a subspace of V.

   (b) A scalar λ is an *eigenvalue* of T if ...

   Solution: A scalar $\lambda \in \mathbb{R}$ is an eigenvalue of T if there exists a nonzero vector $v \in V$ such that

   $$T(v) = \lambda v.$$

   (c) A basis $\mathcal{B}$ of V is an *eigenbasis* for T if ...

   Solution: A basis $\mathcal{B} = (v_1, \dots, v_n)$ of V is an eigenbasis for T if every basis vector is an eigenvector of T (possibly with different eigenvalues); that is, for each $v_j \in \mathcal{B}$ there exists a scalar λ_j such that

   $$T(v_j) = \lambda_j v_j.$$

2. Suppose V is a vector space and let $V \xrightarrow{T} V$ be a linear transformation. Show that if $\mu, \lambda \in \mathbb{R}$ with $\mu \neq \lambda$, then $E_\lambda \cap E_\mu = \{0\}$.

   Solution: Recall that

   $$E_\lambda = \{v \in V : T(v) = \lambda v\}, \qquad E_\mu = \{v \in V : T(v) = \mu v\}.$$

   First note that $0 \in E_\lambda$ and $0 \in E_\mu$, so $0 \in E_\lambda \cap E_\mu$.

   Now let $v \in E_\lambda \cap E_\mu$. Then $v \in E_\lambda$ and $v \in E_\mu$, so

   $$T(v) = \lambda v \quad \text{and} \quad T(v) = \mu v.$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

Thus

$$\lambda v = T(v) = \mu v,$$

so

$$(\lambda - \mu)v = 0.$$

Because $\lambda \neq \mu$, the scalar $\lambda - \mu$ is nonzero. In a vector space, if a nonzero scalar times a vector is 0, then the vector must be 0. Hence $v = 0$.

Therefore, the only vector in both E_λ and E_μ is the zero vector, so

$$E_\lambda \cap E_\mu = \{0\}.$$

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) Suppose V is a vector space. Every linear transformation from V to V has an eigenbasis.

Solution: FALSE.

A linear transformation need not have enough eigenvectors to form a basis.

Counterexample: Let $V = \mathbb{R}^2$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by 90° counterclockwise about the origin. In matrix form (with respect to the standard basis),

$$T(x, y) = (-y, x), \quad [T] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Over $\mathbb{R}$, this transformation has no real eigenvalues and hence no real eigenvectors other than 0. Therefore, there is no eigenbasis for T (since an eigenbasis must consist of eigenvectors).

(b) Suppose V is a vector space and $\mathcal{A} = (v_1, v_2, \ldots, v_k)$ is a linearly independent list of vectors. If $w \in V \setminus \mathrm{Span}(\mathcal{A})$, then $(w, v_1, v_2, \ldots, v_k)$ is linearly dependent.

Solution: FALSE.

Counterexample: Let $V = \mathbb{R}^2$ and let $v_1 = (1, 0)^\top$, so that the list $\mathcal{A} = (v_1)$ is linearly independent. Take $w = (0, 1)^\top$. Then $w \notin \mathrm{Span}(v_1)$.

However, the list

$$(w, v_1) = \left((0, 1)^\top, \, (1, 0)^\top\right)$$

is linearly *independent*.

Thus the list (w, v_1) is not dependent, so the given statement is false.

(c) Suppose $v, w \in \mathbb{R}^n$ are two nonzero vectors and $T \colon \mathbb{R}^n \to \mathbb{R}^n$ has the property that $T(v) = 3v$ and $T(w) = 5w$. Then $v \cdot w = 0$.

Solution: FALSE.

Counterexample: Define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the matrix

$$A = \begin{pmatrix} 3 & 2 \\ 0 & 5 \end{pmatrix}.$$

Let $v = (1,0)^\top$ and $w = (1,1)^\top$. Then

$$T(v) = A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix} = 3\begin{pmatrix}1\\0\end{pmatrix} = 3v,$$

and

$$T(w) = A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}5\\5\end{pmatrix} = 5\begin{pmatrix}1\\1\end{pmatrix} = 5w.$$

So v and w are eigenvectors of T with eigenvalues 3 and 5 respectively. However,

$$v \cdot w = (1,0)\cdot(1,1) = 1\cdot 1 + 0 \cdot 1 = 1 \neq 0.$$

Thus v and w are not orthogonal, contradicting the claim.